Online appendix to “Interpreting Shocks to the Relative Price of Investment with a Two-Sector-Model” by Luca Guerrieri, Dale Henderson, and Jinill Kim

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This appendix provides additional details on the proof of Theorem 1 in the main body of the paper. We first lay down the model to which Theorem 1 refers. Second we derive the conditions for an equilibrium. Third we use the conditions for an equilibrium to derive a set of steady-state conditions. Finally, we provide all the intermediate steps omitted from the main body of the paper to derive Equation 10 and Equation 20 in the paper – transcribed, respectively, as Equation 38 and Equation 49 in this appendix.

1 The model

In period t, the representative household supplies a fixed amount of labor L, and maximizes the intertemporal utility function

$$\max_{Cs,I_s,K_{Ns+1},K_{Ms+1},B_s} E_t \sum_{s=t}^{\infty} \beta^{s-t} \log C_s, \tag{1}$$

subject to the budget constraint

$$W_s L + R_{Ms} K_{Ms} + R_{Ns} K_{Ns} + \rho_{s-1} B_{s-1} = P_{Cs} C_s + P_{Is} I_s + B_s, \tag{2}$$

and subject to the following law of motion for the accumulation of capital

$$K_{Ms+1} + K_{Ns+1} = (1-\delta)(K_{Ms} + K_{Ns}) + I_s. \tag{3}$$

There is complete specialization in the assembly of consumption and investment goods. Hence, $I_t = Y_{Mt}$, and $C_t = Y_{Nt}$. In each sector, perfectly competitive firms minimize production costs to meet demand subject to the technology constraint as reflected in the following Lagrangian problems:

$$\min_{K_{Ms},L_{Ms},P_{Ms}} R_{Ms} K_{Ms} + W_s L_{Ms} + P_{Ms}(Y_{Ms} - K_{Ms}^{\alpha_M} (A_{Ms} L_{Ms})^{1-\alpha_M}), \tag{4}$$

$$\min_{K_{Ns},L_{Ns},P_{Ns}} R_{Ns}K_{Ns} + W_s L_{Ns} + P_{Ns}(Y_{Ns} - K_{Ns}^{\alpha_N} (A_{Ns}L_{Ns})^{1-\alpha_N}). \tag{5}$$

In addition to satisfying the first-order conditions the optimization problems of households and firms, an equilibrium in the model is such that all factor markets and product markets clear.

There is complete specialization across sectors. $I = Y_M$, and $C = Y_N$.

2 Necessary Conditions for an equilibrium

From the household's side, let λ_{Cs} be the Lagrange multiplier on the budget constraint and λ_{Ks} be the Lagrange multiplier on the capital accumulation equation.

From $\frac{\partial}{\partial C_s} = 0$

$$\frac{1}{C_s} - \lambda_{Cs}P_{Cs} = 0. \tag{6}$$

From $\frac{\partial}{\partial I_s} = 0$

$$-\lambda_{Cs}P_{Is} - \lambda_{Ks} = 0. \tag{7}$$

From $\frac{\partial}{\partial K_{Ns}} = 0$

$$\lambda_{Cs+1}\beta E_t R_{Ns+1} + \lambda_{Ks} - E_t\lambda_{Ks+1}\beta(1-\delta) = 0. \tag{8}$$

From $\frac{\partial}{\partial K_{Ms}} = 0$

$$\lambda_{Cs+1}\beta E_t R_{Ms+1} + \lambda_{Ks} - E_t\lambda_{Ks+1}\beta(1-\delta) = 0. \tag{9}$$

From $\frac{\partial}{\partial B_s} = 0$

$$-\lambda_{Cs} + \beta E_t\lambda_{Cs+1}\rho_s = 0. \tag{10}$$

From $\frac{\partial}{\partial \lambda_K}$

$$K_{Ms+1} + K_{Ns+1} = (1-\delta)(K_{Ms} + K_{Ns}) + I_s. \tag{11}$$

From $\frac{\partial}{\partial \lambda_C}$

$$W_s L + R_{Ms}K_{Ms} + R_{Ns}K_{Ns} + \rho_{s-1}B_{s-1} = P_{Cs}C_s + P_{Is}I_s + B_s. \tag{12}$$

From complete specialization

$$I_s = Y_{M_s}, \tag{13}$$

and

$$C_s = Y_{Ns.} \tag{14}$$

And also

$$P_{Is} = P_{Ms} \tag{15}$$

and

$$P_{Cs} = P_{Ns} \tag{16}$$

From the firms' problem using $\frac{\partial}{\partial K_{Ms}} = 0$

$$R_{Ms} - P_{Ms}\alpha_M K_{Ms}^{\alpha_M - 1} \left(A_M L_{Ms}\right)^{1-\alpha_M} = 0 \tag{17}$$

$$R_{Ms} - P_{Ms}\alpha_M \frac{Y_{Ms}}{K_{Ms}} = 0 \tag{18}$$

From $\frac{\partial}{\partial L_{Ms}} = 0$

$$W_s - P_{Ms}\left(1-\alpha_M\right) K_{Ms}^{\alpha_M} \left(A_M L_{Ms}\right)^{-\alpha_M} A_M = 0 \tag{19}$$

$$W_s - P_{Ms}\left(1-\alpha_M\right) \frac{Y_{Ms}}{L_{Ms}} = 0 \tag{20}$$

From $\frac{\partial}{\partial P_{Ms}} = 0$

$$Y_{Ms} - K_{Ms}^{\alpha_M} \left(A_M L_{Ms}\right)^{1-\alpha_M} \tag{21}$$

From $\frac{\partial}{\partial K_{Ns}} = 0$

$$R_{Ns} - P_{Ns}\alpha_N K_{Ns}^{\alpha_N - 1} \left(A_N L_{Ns}\right)^{1-\alpha_N} = 0 \tag{22}$$

$$R_{Ns} - P_{Ns}\alpha_N \frac{Y_{Ns}}{K_{Ns}} = 0 \tag{23}$$

From $\frac{\partial}{\partial L_{Ns}} = 0$

$$W_s - P_{Ns}\left(1-\alpha_N\right) K_{Ns}^{\alpha_N}\left(A_N L_{Ns}\right)^{-\alpha_N} A_N = 0 \tag{24}$$

$$W_s - P_{Ns}\left(1-\alpha_N\right)\frac{Y_{Ns}}{L_{Ns}} = 0 \tag{25}$$

From $\frac{\partial}{\partial P_{Ns}} = 0$

$$Y_{Ns} - K_{Ns}^{\alpha_N}\left(A_N L_{Ns}\right)^{1-\alpha_N} \tag{26}$$

3 Derivation of some steady-state restrictions

Equation I)

Work on $\frac{\partial}{\partial K_{Ns}} = 0$. From $\frac{1}{C_s} - \lambda_{Cs} P_{Cs} = 0$, and using full specialization

$$\frac{1}{P_N C} = \lambda_C.$$

Furthermore, with $-\lambda_{Cs} P_{Ms} = \lambda_{Ks}$, which can be expressed as $-\frac{P_{Ms}}{P_{Ns} C_s} = \lambda_{Ks}$ one obtains:

$$\beta\frac{R_N}{P_N C} - \frac{P_M}{P_N C} + \beta\frac{P_M}{P_N C}(1-\delta) = 0.$$

Equation II)

Combining $\frac{\partial}{\partial K_{Ns}} = 0$ and $\frac{\partial}{\partial K_{Ms}} = 0$

$$R_N = R_M.$$

Equation III)

From the firms' problem, using $\frac{\partial}{\partial K_{\ Ms}} = 0$

$$R_M = P_M \alpha_M \frac{Y_M}{K_M}.$$

Equation IV)

From the firms' problem, using $\frac{\partial}{\partial L_{Ms}} = 0$

$$W = P_M \left(1 - \alpha_M\right) \frac{Y_M}{L_M}.$$

Equation V)

From the firms' problem, using $\frac{\partial}{\partial K_{\ Ns}} = 0$

$$R_{Ns} = P_{Ns} \alpha_N \frac{Y_{Ns}}{K_{Ns}}.$$

Equation VI)

From the firms' problem, using $\frac{\partial}{\partial L_{Ms}} = 0$

$$W = P_N \left(1 - \alpha_N\right) \frac{Y_N}{L_N}.$$

Equation VII)

Using the production technology for sector M,

$$Y_M = K_M^{\alpha_M} \left(A_M L_M\right)^{1-\alpha_M}$$

Equation VIII)

Using the production technology for sectory N,

$$Y_N = K_N^{\alpha_N} \left(A_N L_N\right)^{1-\alpha_N}$$

Equation IX)

From full specialization

$$Y_M = I$$

Equation X)

From full specialization

$$Y_N = C$$

Equation XI)

From market clearing

$$L_M + L_N = L$$

Equation XII)

Using the capital accumulation equation, $K_{Ms+1} + K_{Ns+1} = (1-\delta)(K_{Ms} + K_{Ns}) + I_s$, with complete specialization

$$\delta(K_M + K_N) = I, \tag{27}$$

$$K_M + K_N = \frac{1}{\delta} Y_M. \tag{28}$$

4 Proof of Theorem 1: Part 1, The Long-Run Response of Relative Prices

From I), multiplying both sides by $P_N C$

$$\beta R_N - P_M + \beta P_M (1-\delta) = 0. \tag{29}$$

Combing the equation above with II) in Table 1

$$R_M = R_N = \frac{1}{\beta} P_M \left(1 - \beta(1-\delta)\right) \tag{30}$$

Solve VII) in Table 1 for K_M

$$K_M = \left(\frac{Y_M}{(A_M L_M)^{1-\alpha_M}} \right)^{\frac{1}{\alpha_M}} \tag{31}$$

and substitute it into III) in Table 1 to yield:

$$R_M = P_M \alpha_M \frac{Y_M}{\left(\frac{Y_M}{(A_M L_M)^{1-\alpha_M}} \right)^{\frac{1}{\alpha_M}}}$$

which simplifies to

$$\frac{\left(\frac{Y_M}{(A_M L_M)^{1-\alpha_M}}\right)^{\frac{1}{\alpha_M}}}{Y_M} = \alpha_M \frac{P_M}{R_M}$$

$$\frac{Y_M^{\frac{1-\alpha_M}{\alpha_M}}}{(A_M L_M)^{\frac{1-\alpha_M}{\alpha_M}}} = \alpha_M \frac{P_M}{R_M}$$

$$\frac{Y_M}{L_M} = A_M \left(\alpha_M \frac{P_M}{R_M}\right)^{\frac{\alpha_M}{1-\alpha_M}} . \tag{32}$$

Analogously from V) and VIII) in Table 1, we obtain:

$$\frac{Y_N}{L_N} = A_N \left(\alpha_N \frac{P_N}{R_N}\right)^{\frac{\alpha_N}{1-\alpha_N}} . \tag{33}$$

Next, combine IV) and VI) in Table 1 to yield:

$$\frac{P_M}{P_N} = \frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{Y_N}{L_N} \frac{L_M}{Y_M}. \tag{34}$$

Substituting equations 30, 32, and 33 into equation 34, one can solve for $\frac{P_M}{P_N}$ in terms of parameters and the levels of sector-specific technology A_M and A_N:

$$\frac{P_M}{P_N} = \frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{A_N \left(\alpha_N \frac{P_N}{P_M} \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{A_M \left(\alpha_M \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_M}{1-\alpha_M}}}$$

Now solving for $\frac{P_M}{P_N}$

$$\frac{P_M}{P_N} = \left(\frac{P_N}{P_M}\right)^{\frac{\alpha_N}{1-\alpha_N}} \frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{A_N \left(\alpha_N \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{A_M \left(\alpha_M \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_M}{1-\alpha_M}}} \tag{35}$$

and simplifying further

$$\left(\frac{P_M}{P_N}\right)^{1+\frac{\alpha_N}{1-\alpha_N}} = \frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{A_N\left(\alpha_N \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{A_M\left(\alpha_M \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_M}{1-\alpha_M}}} \tag{36}$$

$$\left(\frac{P_M}{P_N}\right)^{\frac{1}{1-\alpha_N}} = \frac{A_N}{A_M}\frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{\left(\alpha_N \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{\left(\alpha_M \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_M}{1-\alpha_M}}} \tag{37}$$

$$\frac{P_M}{P_N} = \psi_1\left(\frac{A_N}{A_M}\right)^{1-\alpha_N}, \quad \text{where } \psi_1 = \left(\frac{(1-\alpha_N)}{(1-\alpha_M)} \frac{\left(\alpha_N \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_N}{1-\alpha_N}}}{\left(\alpha_M \frac{1}{\frac{1}{\beta}(1-\beta(1-\delta))}\right)^{\frac{\alpha_M}{1-\alpha_M}}}\right)^{1-\alpha_N}. \tag{38}$$

Thus, equiproportionate changes in technology in the two production sectors M and N will not affect relative prices. Variation in relative prices at the sectoral level is a precondition for variation in relative prices at the level of final goods. Thus, the result derived here extends to the model in the main body of the paper with incomplete sectoral specialization in the assembly of consumption and investment goods, as reflected in the numerical simulations.

5 Proof of Theorem 1: Part 2, The Long-Run Response of Labor Productivity

Define labor productivity as:

$$\frac{Y_{Mt}+Y_{Nt}}{L} = \frac{Y_{Mt}}{L_{Mt}}\frac{L_{Mt}}{L} + \frac{Y_{Nt}}{L_{Nt}}\frac{L_{Nt}}{L}. \tag{39}$$

First work on obtaining $\frac{L_{Mt}}{L}$ and $\frac{L_{Nt}}{L}$. Using V, VIII, 30 and III, VII, 30 one can obtain, respectively:

$$\frac{K_M}{Y_M} = \frac{\alpha_M}{(1-\beta(1-\delta))}, \tag{40}$$

$$\frac{K_N}{Y_N} = \frac{\alpha_N}{(1-\beta(1-\delta))}\frac{P_N}{P_M}. \tag{41}$$

$\frac{K_N}{Y_N}$ can be related to technology levels through 38. From XII, one has that:

$$\frac{K_N}{Y_N}\frac{Y_N}{Y_M} + \frac{K_M}{Y_M} = \frac{1}{\delta}, \tag{42}$$

which can be used with to 40 and 41 to solve for $\frac{Y_N}{Y_M}$:

$$\frac{Y_N}{Y_M} = \psi_2 \left(\frac{A_N}{A_M}\right)^{1-\alpha_N}, \quad \text{where} \quad \psi_2 = \psi_1 \left(\frac{(1-\beta(1-\delta))}{\delta\alpha_N} - \frac{\alpha_M}{\alpha_N}\right) \tag{43}$$

Combining IV, VI, and XI, one obtains:

$$\frac{L_M}{L} = \frac{(1-\alpha_M)P_{Mt}Y_{Mt}}{(1-\alpha_N)P_{Nt}Y_{Nt} + (1-\alpha_M)P_{Mt}Y_{Mt}}, \tag{44}$$

which can be expressed as a function of parameters and technology levels as:

$$\frac{L_M}{L} = \frac{(1-\alpha_M)\psi_1}{(1-\alpha_M)\psi_1 + (1-\alpha_N)\psi_2}, \tag{45}$$

and since $L_N + L_M = L$ once can see that:

$$\frac{L_N}{L} = \frac{(1-\alpha_N)\psi_2}{(1-\alpha_M)\psi_1 + (1-\alpha_N)\psi_2}. \tag{46}$$

Next work on $\frac{Y_{Mt}}{L_{Mt}}$ and on $\frac{Y_{Nt}}{L_{Nt}}$. Combining equations 32 and 33 with equation 30 yields:

$$\frac{Y_M}{L_M} = A_M \left(\frac{\alpha_M}{(1-\beta(1-\delta))}\right)^{\frac{\alpha_M}{1-\alpha_M}}, \tag{47}$$

$$\frac{Y_N}{L_N} = A_N \left(\frac{\alpha_N}{(1-\beta(1-\delta))}\frac{P_N}{P_M}\right)^{\frac{\alpha_N}{1-\alpha_N}}. \tag{48}$$

Summing up, remembering that $\frac{P_M}{P_N} = \psi_1 \left(\frac{1}{A}\right)^{1-\alpha_N}$, one can see that at constant prices:

$$\begin{aligned}
\frac{Y_M + Y_N}{L} \quad &= \quad \frac{Y_M}{L_M}\frac{L_M}{L} + \frac{Y_N}{L_N}\frac{L_N}{L} = \\
&A_M \left(\frac{\alpha_M}{(1-\beta(1-\delta))}\right)^{\frac{\alpha_M}{1-\alpha_M}} \frac{(1-\alpha_M)\psi_1}{(1-\alpha_M)\psi_1 + (1-\alpha_N)\psi_2} \\
&+ A_M^{\alpha_N} A_N^{1-\alpha_N} \left(\frac{\alpha_N}{\psi_1\,(1-\beta(1-\delta))}\right)^{\frac{\alpha_N}{1-\alpha_N}} \frac{(1-\alpha_N)\psi_2}{(1-\alpha_M)\psi_1 + (1-\alpha_N)\psi_2}.
\end{aligned} \tag{49}$$

Notice that Fisher defined aggregate labor productivity in terms of consumption units (i.e., $\frac{Y_{Mt}}{L_{Mt}}\frac{L_{Mt}}{L}\frac{P_M}{P_N} + \frac{Y_N}{L_N}\frac{L_N}{L}$) rather than at constant prices. Even under that alternative aggregation, labor productivity remains a log-linear function of both shocks. Taken together, Equations 38 and 49 prove Theorem 1.

Table 1: Steady State Restrictions

I)	$\beta\frac{R_N}{P_N C} - \frac{P_M}{P_N C} + \beta\frac{P_M}{P_N C}(1-\delta) = 0$	II)	$R_{Nt} = R_{Mt}$
III)	$R_M = P_M \alpha_M \frac{Y_M}{K_M}$	IV)	$W = P_M(1-\alpha_M)\frac{Y_M}{L_M}$
V)	$R_N = P_N \alpha_N \frac{Y_N}{K_N}$	VI)	$W = P_N(1-\alpha_N)\frac{Y_N}{L_N}$
VII)	$Y_M = K_M^{\alpha_M}(AGL_M)^{1-\alpha_M}$	VIII)	$Y_N = K_N^{\alpha_N}(GL_N)^{1-\alpha_N}$
IX)	$Y_M = I$	X)	$Y_N = C$
XI)	$L_M + L_N = L$	XII)	$K_M + K_N = \frac{1}{\delta} Y_M$